

IFP-UNC-523  
TAR-UNC-065  
gr-qc/9802027

# CURVATURE-BASED HYPERBOLIC SYSTEMS FOR GENERAL RELATIVITY

YVONNE CHOQUET-BRUHAT  
Gravitation et Cosmologie Relativiste  
t.22-12, Un. Paris VI  
Paris 75252 France

JAMES W. YORK, JR. and ARLEN ANDERSON  
Department of Physics and Astronomy  
Univ. North Carolina  
Chapel Hill NC 27599-3255 USA

Feb. 6, 1998

## **Abstract**

We review curvature-based hyperbolic forms of the evolution part of the Cauchy problem of General Relativity that we have obtained recently. We emphasize first order symmetrizable hyperbolic systems possessing only physical characteristics.

# 1 Introduction

Application of general relativity to fully three dimensional problems in astrophysics and cosmology has provided a driving force motivating further study and reformulation of Einstein's equations in 3+1 form. The vital role of the four constraint equations in setting up initial data for numerical evolution and in carrying out mathematical studies of gravitational field configurations has long been recognized. It led in the 1970's to a general, mathematically rigorous, and useful formulation of the constraints. (See, for example, the review [1].) The standard evolution equations for the spatial metric  $\mathbf{g}$  and the extrinsic curvature  $\mathbf{K}$  were already known from a straightforward decomposition of the spacetime Ricci tensor (cf. the cases of zero shift [2], arbitrary shift [3], and synthesis from the spacetime viewpoint and further developments [4]). It was widely believed that these equations (first order in time derivatives; second order in space derivatives) were adequate for applications. However, they do not constitute a hyperbolic system leading to a proof of causal evolution in local Sobolev spaces of  $\mathbf{g}$  and  $\mathbf{K}$  into Einsteinian spacetime. Hence, further study of evolution equations is essential.

In this paper, we review methods that we have used recently to obtain hyperbolic systems for the evolution of  $(\mathbf{g}, \mathbf{K})$  with only physical characteristics and which propagate curvature along the physical light cone. We emphasize first order symmetrizable hyperbolic systems.

## 2 Einstein-Ricci Hyperbolic System

What we may call the "Einstein-Ricci" (ER) system is a spatially covariant hyperbolic formulation of the Einstein equations constructed from first derivatives of the spacetime Ricci tensor. The ER equations are obtained from the 3+1 form of the metric

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (1)$$

where  $N > 0$  is the lapse,  $\beta^i$  is the shift, and the properly Riemannian metric  $g_{ij}$  is the spatial metric of a spacelike slice  $\Sigma$ . ( $\Sigma$  is understood to be a generic spacelike leaf of a foliation of a globally hyperbolic spacetime: locally,  $t=\text{const.}$ ) It is convenient to work in the cobasis  $\theta^0 = dt$ ,  $\theta^i = dx^i + \beta^i dt$ , with  $d\theta^\alpha = -\frac{1}{2}C^\alpha_{\beta\gamma}\theta^\beta \wedge \theta^\gamma$  and only  $C^i_{0j} = -C^i_{j0} = \partial_j \beta^i \neq 0$ . The connection one-forms  $\omega^\alpha_\beta = \gamma^\alpha_{\beta\gamma}\theta^\gamma$  are given by the connection coefficients

$$\gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + g^{\alpha\delta} C^\epsilon_{\delta(\beta} g_{\gamma)\epsilon} - \frac{1}{2} C^\alpha_{\beta\gamma}, \quad (2)$$

where  $\Gamma$  denotes a Christoffel symbol. In particular, the extrinsic curvature of  $\Sigma$  is  $K_{ij} \equiv -N\gamma^0_{ij}$ , or, upon evaluation of  $\gamma^0_{ij}$ ,

$$\hat{\partial}_0 g_{ij} = -2NK_{ij}, \quad (3)$$

where  $\hat{\partial}_0 = \partial_t - \mathcal{L}_\beta$  evolves  $t$ -dependent spatial objects in the direction orthogonal to  $\Sigma$ , and  $\mathcal{L}_\beta$  is the Lie derivative in  $\Sigma$  along the shift vector. The relation (3) serves as the evolution equation for  $g_{ij}$ . Letting an overbar signify spatial objects, we note for future reference that  $\bar{g}_{ij} = g_{ij}$ ,  $\bar{g}^{ij} = g^{ij}$ , and  $\gamma^i_{jk} = \Gamma^i_{jk} = \bar{\Gamma}^i_{jk}$ . For completeness, we also state  $\gamma^i_{00} = N\bar{\nabla}^i N$ ,  $\gamma^0_{00} = N^{-1}\hat{\partial}_0 N$ ,  $\gamma^0_{0i} = \gamma^0_{i0} = N^{-1}\bar{\nabla}_i N$ ,  $\gamma^k_{0i} = -NK^k_i$ , and  $\gamma^k_{i0} = \gamma^k_{0i} + \partial_i \beta^k$ .

The 3+1 decomposition of the spacetime Riemann tensor is given, for example, in Ref. [4]. Likewise, the Ricci tensor is given identically by

$$R_{ij} \equiv \bar{R}_{ij} - N^{-1}\hat{\partial}_0 K_{ij} + HK_{ij} - 2K_{ik}K^k_j - N^{-1}\bar{\nabla}_i \bar{\nabla}_j N, \quad (4)$$

$$R_{0i} \equiv N\bar{\nabla}^j (Hg_{ij} - K_{ij}), \quad (5)$$

$$R_{00} \equiv N\hat{\partial}_0 H - N^2 K_{mk} K^{mk} + N\bar{\nabla}^k \bar{\nabla}_k N, \quad (6)$$

where  $H \equiv K^k_k \equiv K$ . The heart of the ER system [5]-[7] is defined by

$$\Omega_{ij} \equiv \hat{\partial}_0 R_{ij} - 2\bar{\nabla}_{(i} R_{j)0}. \quad (7)$$

Upon working out this identity using (4)-(6) and substituting the Einstein equations  $R_{\alpha\beta} = \rho_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T^\gamma_\gamma)$ , with  $G = c = 1$ , we obtain the equation

$$-N\hat{\square}K_{ij} = J_{ij} - S_{ij} + \Omega_{ij}, \quad (8)$$

where the physical wave operator for arbitrary shift is  $\hat{\square} \equiv -N^{-1}\hat{\partial}_0 N^{-1}\hat{\partial}_0 + \bar{\nabla}^k \bar{\nabla}_k$ . We study the *vacuum* case  $\Omega_{ij} = 0$  here. In general, the value of  $\Omega_{ij}$  is defined by matter fields.

The detailed form of the right hand side of (8) can be found in Refs. [5]-[7]; the present conventions are those in Refs. [6],[7]. Here we point out that  $J_{ij} = J_{ij}[2 \text{ in } \mathbf{g}; 2 \text{ in } N; 1 \text{ in } \mathbf{K}]$ . (The numbers in the bracket indicate the highest order derivatives that occur.)

The slicing-dependent term  $S_{ij}$  is given by

$$S_{ij} = -N^{-1}\bar{\nabla}_i \bar{\nabla}_j (\hat{\partial}_0 N + N^2 H). \quad (9)$$

We observe that the term  $\bar{\nabla}_i \bar{\nabla}_j H = g^{mk} \bar{\nabla}_i \bar{\nabla}_j K_{mk}$  would spoil the hyperbolicity of (8); therefore,  $S_{ij}$  must be set equal to a functional involving fewer than two derivatives of  $K_{ij}$ . Notice that

$$\hat{\partial}_0 N + N^2 H \equiv -N^3 \gamma^0 \equiv g^{1/2} \hat{\partial}_0 (g^{-1/2} N) \equiv g^{1/2} \hat{\partial}_0 \alpha, \quad (10)$$

where  $g = \det g_{ij} > 0$ ,  $\gamma^0 \equiv {}^{(4)}g^{\alpha\beta} \gamma^0_{\alpha\beta}$ , and  $\alpha \equiv g^{-1/2} N$  is a lapse function of weight -1, first introduced in the “algebraic gauge” [8] and subsequently as a “proper-time gauge” [9] and in connection with the “new variables” program for general relativity in Hamiltonian form [10], where  $\alpha = \underline{N}$  is called the “densitized lapse.” The algebraic gauge has a simple relationship to “harmonic time slicing,”  $\gamma^0 = 0$ , as we see in (10).

One most easily deals with  $S_{ij}$  by freely specifying  $\alpha(t, x) > 0$ . (The shift  $\beta^i(t, x)$  is also arbitrary.) A “gauge source”  $f(t, x)$  can be used, as observed by Friedrich (see [11] for references); here,  $f(t, x) = \hat{\partial}_0 \log \alpha$ . Then the first term in (10) yields a general harmonic time slicing equation for  $N$ , namely

$$\hat{\partial}_0 N + N^2 H = N f(t, x), \quad (11)$$

or equivalently,

$$\hat{\partial}_0 (g^{-1/2} N) = g^{-1/2} N f(t, x). \quad (12)$$

In this case, the system composed of (12) and the third-order equation resulting from combining (3) and (8), whose principal term is  $\square \partial_0 g_{ij}$ , is quasi-diagonal hyperbolic [12] for the unknowns  $\mathbf{g}$  and  $g^{-1/2} N$ . Therefore, the “third-order” ER equations have a unique, well-posed solution in a suitable Sobolev function space (local in time and in space; global in space can be obtained as well), given appropriate initial data [5, 13].

We wish to verify that every solution of the third order ER system, equivalently  $\{(3), (8), (12) \text{ or } (11)\}$ , is a solution of the Einstein equations given suitable initial data. Thus, suppose  $\mathbf{g}$ ,  $\mathbf{K}$ , and  $N$  satisfy these equations. Then, by using the contracted Bianchi identities  $\nabla^\beta G_{\beta\alpha} \equiv 0$ , and these ER equations, it follows that  $G_{\beta\alpha} = 0$  if  $\mathbf{g}$  and  $\mathbf{K}$  satisfy initially the usual momentum and Hamiltonian constraints ( $\mathcal{C}_i \equiv -2N^{-1}R_{0i} = 0$ ,  $\mathcal{C} \equiv 2G^0_0 = 0$ ), if  $\hat{\partial}_0 K_{ij}$  satisfies  $R_{ij} = 0$  initially, and if initial data  $\{\alpha > 0, \beta^i\}$  or  $\{N > 0, \beta^i\}$  are given [5, 6]. Conversely, because every globally hyperbolic metric solution of Einstein’s equations can be given in a harmonic time slicing [1], it follows that all globally hyperbolic solutions of Einstein’s equations can be reached by solving this Einstein-Ricci system.

The ER system as given holds for any spacetime dimension. For *four* dimensions, it can be written in first-order symmetrizable hyperbolic (FOSH) form. To put a system in first-order form, one introduces auxiliary variables in place of derivatives of the

fundamental variables and adds additional equations to describe the evolution of these new variables. The crux is whether this process terminates. Generally existence of a higher order wave equation on the fundamental variables is necessary to halt the process. Though we have a wave equation for  $\mathbf{K}$ , presence of second spatial derivatives of  $\mathbf{g}$  and  $N$  in  $J_{ij}$  prevent the reduction to first order form from being obvious. However, the second derivatives of  $\mathbf{g}$  occur only in  $\bar{R}_{ij}$  and  $\bar{R}_{ijkl}$  terms; the latter, in three space dimensions, can be reduced to  $\bar{R}_{ij}$ , which in turn can be eliminated from  $J_{ij}$  by using (4). First derivatives of  $\mathbf{g}$  are handled by an evolution equation [5, 7] for the Christoffel symbol  $\bar{\Gamma}$  (inadvertently omitted in [6]).

On the other hand, to handle the second derivatives of  $N$ , one first finds a wave equation for  $N$ . Then, by applying  $\bar{\nabla}_i$  to it, a wave equation for  $\bar{\nabla}_i N$ , or  $a_i = \bar{\nabla}_i \log N$ , is obtained. From this point, reduction to first order form is straightforward [5]-[7]. [The wave equation for  $N$  follows from applying  $\hat{\partial}_0$  to (11) to get an equation with  $\hat{\partial}_0 \hat{\partial}_0 N$  and  $\hat{\partial}_0 H$ . Using (6), we then find a nonlinear wave equation that can be written in terms of  $\hat{\square} N$ . Then  $\bar{\nabla}_i \hat{\square} N$  gives the wave equation  $\hat{\square} a_i$ . See Refs. [5], [7] for details.] One then shows that the first order system is symmetrizable and hyperbolic.

In this system, the spacetime metric  $(N, g_{ij})$  *evolves* at zero speed along the direction  $\hat{\partial}_0$  orthogonal to  $t=\text{constant}$ . Quantities with dimensions of curvature *propagate* at speed 1 (“ $c$ ”) along the physical light cone. There are *no* unphysical speeds or directions as there are in many of the FOSH systems that have so far appeared in the literature [11, 14, 15]. Of course, we except the FOSH form of the ER system [5]-[7] (described here) and the mathematically equivalent, but more transparent, “Einstein-Bianchi” [16, 17] (EB) system presented in the next section. Having only physical characteristics is essential for an ideal match of physics and mathematics. In practice, it is also useful: for example, consider obtaining the gravitational radiation content of a numerical Cauchy evolution on a finite grid. The radiation is propagating at light speed and is either “extracted” [18] or “evolved to null infinity” [19] based on field values at a finite distance from the source. We know that gravitational radiation is curvature propagating at light speed and that is what the ER system describes. The description is even more explicit and transparent in the EB system.

A “fourth order” ER system also exists [20]. One forms the expression

$$\hat{\partial}_0 \Omega_{ij} + \bar{\nabla}_i \bar{\nabla}_j R_{00} \quad (13)$$

to obtain it. It has a well-posed Cauchy problem and is hyperbolic (“non-strict” [21]) for any  $N > 0$  and  $\beta^i$ . It has been applied to the non-linear perturbative regime of high frequency wave propagation [22, 23] and has been used to obtain an elegant derivation of gauge-invariant perturbation theory for the Schwarzschild metric [24].

### 3 Einstein-Bianchi Hyperbolic System

The “Einstein-Bianchi” (EB) system discussed next was given first in [16] and with energy estimates and full mathematical detail in [17]. It is similar to an analogous system, obtained by H. Friedrich [11], that is based on the Weyl tensor and is causal but with additional unphysical characteristics. Recall that the Riemann tensor satisfies the Bianchi identities

$$\nabla_\alpha R_{\beta\gamma,\lambda\mu} + \nabla_\gamma R_{\alpha\beta,\lambda\mu} + \nabla_\beta R_{\gamma\alpha,\lambda\mu} \equiv 0, \quad (14)$$

where we use a comma to stress the two separate antisymmetric index pairs (*not* to indicate partial differentiation). These identities imply by contraction and use of the symmetries of the Riemann tensor

$$\nabla_\alpha R^\alpha_{\beta,\lambda\mu} + \nabla_\mu R_{\beta\lambda} - \nabla_\lambda R_{\beta\mu} \equiv 0, \quad (15)$$

where the Ricci tensor is defined by

$$R^\alpha_{\beta,\alpha\mu} = R_{\beta\mu}.$$

If the Ricci tensor satisfies the Einstein equations

$$R_{\alpha\beta} = \rho_{\alpha\beta}, \quad (16)$$

then we have

$$\nabla_\alpha R^\alpha_{\beta,\lambda\mu} = \nabla_\lambda \rho_{\beta\mu} - \nabla_\mu \rho_{\beta\lambda}. \quad (17)$$

Equation (14) with  $\{\alpha\beta\gamma\} = \{ijk\}$  and (17) with  $\beta = 0$  contain only derivatives of the Riemann tensor tangent to  $\Sigma$ ; hence, we consider these equations as constraints (“Bianchi constraints”). We shall consider the remaining equations in (14) and (17) as applying to a double two-form  $A_{\alpha\beta,\lambda\mu}$ , which is simply a spacetime tensor antisymmetric in its first and last pairs of indices. We do *not* suppose *a priori* a symmetry between the two pairs of antisymmetric indices. These “Bianchi equations” are

$$\nabla_0 A_{hk,\lambda\mu} + \nabla_k A_{0h,\lambda\mu} + \nabla_h A_{k0,\lambda\mu} = 0, \quad (18)$$

$$\nabla_0 A^0_{i,\lambda\mu} + \nabla_h A^h_{i,\lambda\mu} = \nabla_\lambda \rho_{i\mu} - \nabla_\mu \rho_{i\lambda} \equiv J_{i,\lambda\mu}, \quad (19)$$

where the pair  $[\lambda\mu]$  is either  $[0j]$  or  $[jl]$ . We next introduce following Bel [25] two “electric” and two “magnetic” space tensors associated with the double two-form  $\mathbf{A}$ ,

in analogy to the electric and magnetic vectors associated with the electromagnetic two-form  $\mathbf{F}$ . That is, we define the “electric” tensors by

$$E_{ij} \equiv A^0_{i,0j} = -N^{-2}A_{0i,0j}, \quad (20)$$

$$D_{ij} \equiv \frac{1}{4}\eta_{ihk}\eta_{jlm}A^{hk,lm},$$

while the “magnetic” tensors are given by

$$\begin{aligned} H_{ij} &\equiv \frac{1}{2}N^{-1}\eta_{ihk}A^{hk}_{,0j}, \\ B_{ji} &\equiv \frac{1}{2}N^{-1}\eta_{ihk}A_{0j,}{}^{hk}. \end{aligned} \quad (21)$$

In these formulae,  $\eta_{ijk}$  is the volume form of the space metric  $\mathbf{g}$ . We note that: (1) If the double two-form  $\mathbf{A}$  is symmetric with respect to its two pairs of antisymmetric indices, then  $E_{ij} = E_{ji}$ ,  $D_{ij} = D_{ji}$ , and  $H_{ij} = B_{ji}$ . (2) If  $\mathbf{A}$  is a symmetric double two-form such that  $A_{\alpha\beta} \equiv A^\lambda_{\alpha,\lambda\beta} = cg_{\alpha\beta}$ , then  $H_{ij} = H_{ji} = B_{ji} = B_{ij}$  and  $E_{ij} = D_{ij}$ . Property (1) is obvious and (2) follows from the “Lanczos identity.” [26] (We note that  $\eta_{0ijk} = N\eta_{ijk}$  relates the spacetime and space volume forms in using the Lanczos identity.)

In order to extend the treatment to the non-vacuum case and to avoid introducing unphysical characteristics in the solution of the Bianchi equations, we will keep as independent unknowns the four tensors  $\mathbf{E}$ ,  $\mathbf{D}$ ,  $\mathbf{B}$ , and  $\mathbf{H}$ , which will not be regarded necessarily as symmetric. The symmetries will be imposed eventually on the initial data and shown to be conserved by evolution.

We now express the Bianchi equations in terms of the time dependent space tensors  $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{D}$ , and  $\mathbf{B}$ . We also express spacetime covariant derivatives  $\nabla$  of the spacetime tensor  $\mathbf{A}$  in terms of space covariant derivatives  $\bar{\nabla}$  and time derivatives  $\hat{\partial}_0$  of  $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{D}$ , and  $\mathbf{B}$  by using the connection coefficients in 3+1 form as given earlier.

The first Bianchi equation (18) with  $[\lambda\mu] = [0j]$  has the form

$$\nabla_0 A_{hk,0j} + \nabla_k A_{0h,0j} - \nabla_h A_{0k,0j} = 0. \quad (22)$$

A calculation yields (22) in the form

$$\hat{\partial}_0(\eta^i_{hk}H_{ij}) + 2N\bar{\nabla}_{[h}E_{k]j} + (L_1)_{hk,j} = 0, \quad (23)$$

$$\begin{aligned} (L_1)_{hk,j} &\equiv NK^l_j\eta^i_{hk}H_{il} + 2(\bar{\nabla}_{[h}N)E_{k]j} + 2N\eta^i_{lj}K^l_{[k}B_{h]i} \\ &\quad - (\bar{\nabla}^l N)\eta^i_{hk}\eta^m_{lj}D_{im}. \end{aligned} \quad (24)$$

The second Bianchi equation (19), with  $[\lambda\mu] = [0j]$ , has the form

$$\nabla_0 A^0_{i,0j} + \nabla_h A^h_{i,0j} = J_{i,0j}, \quad (25)$$

where  $\mathbf{J}$  is zero in vacuum. A calculation similar to the one above yields for the second Bianchi equation

$$\hat{\partial}_0 E_{ij} - N\eta^{hl}_i \bar{\nabla}_h H_{lj} + (L_2)_{ij} = J_{i,0j}, \quad (26)$$

$$\begin{aligned} (L_2)_{ij} \equiv & -N(\text{Tr}\mathbf{K})E_{ij} + NK^k_j E_{ik} + 2NK^k_i E_{kj} \\ & -(\bar{\nabla}_h N)\eta^{hl}_i H_{lj} + NK^k_h \eta^{lh}_i \eta^m_{kj} D_{lm} + (\bar{\nabla}^k N)\eta^l_{kj} B_{il}. \end{aligned} \quad (27)$$

The non-principal terms in the first two Bianchi equations (23) and (26) are linear in  $\mathbf{E}$ ,  $\mathbf{D}$ ,  $\mathbf{B}$ , and  $\mathbf{H}$ , as well as in the other geometrical elements  $N\mathbf{K}$  and  $\bar{\nabla}N$ . The characteristic matrix of the principal terms is symmetrizable. The unknowns  $E_{i(j)}$  and  $H_{i(j)}$ , with fixed  $j$  and  $i = 1, 2, 3$  appear only in the equations with given  $j$ . The other unknowns appear in non-principal terms. The characteristic matrix is composed of three blocks around the diagonal, each corresponding to one given  $j$ .

The  $j^{\text{th}}$  block of the characteristic matrix in an orthonormal frame for the space metric  $\mathbf{g}$ , with unknowns listed horizontally and equations listed vertically, ( $j$  is suppressed) is given by

$$\begin{array}{l} (26)_1 \\ (26)_2 \\ (26)_3 \\ (23)_{23} \\ (23)_{31} \\ (23)_{12} \end{array} \begin{pmatrix} E_1 & E_2 & E_3 & H_1 & H_2 & H_3 \\ \xi_0 & 0 & 0 & 0 & N\xi_3 & -N\xi_2 \\ 0 & \xi_0 & 0 & -N\xi_3 & 0 & N\xi_1 \\ 0 & 0 & \xi_0 & N\xi_2 & -N\xi_1 & 0 \\ 0 & -N\xi_3 & N\xi_2 & \xi_0 & 0 & 0 \\ N\xi_3 & 0 & -N\xi_1 & 0 & \xi_0 & 0 \\ -N\xi_2 & N\xi_1 & 0 & 0 & 0 & \xi_0 \end{pmatrix}. \quad (28)$$

This matrix is symmetric and its determinant is the characteristic polynomial of the  $\mathbf{E}$ ,  $\mathbf{H}$  system. It is given by

$$-N^6(\xi_0\xi^0)(\xi_\alpha\xi^\alpha)^2. \quad (29)$$

The characteristic matrix is symmetric in an orthonormal space frame and the timelike direction defined by  $\hat{\partial}_0$  has a coefficient matrix  $T_0$  that is positive definite (here  $T_0$  is the unit matrix). Therefore, the first order system is symmetrizable hyperbolic. We will not compute the symmetrized form explicitly here.

The second pair of Bianchi equations is obtained from (18) and (19) with  $[\lambda\mu] = [lm]$ . We obtain from (18)

$$\hat{\partial}_0(\eta^i_{hk}\eta^j_{lm}D_{ij}) + 2N\eta^j_{lm}\bar{\nabla}_{[k}B_{h]j} + (L_3)_{hk,lm} = 0, \quad (30)$$

$$\begin{aligned} (L_3)_{hk,lm} \equiv & 2N\eta^n_{j[m}K^j_{l]}\eta^i_{hk}D_{in} + 2\eta^j_{lm}(\bar{\nabla}_{[k}N)B_{h]j} \\ & + 2NK_{l[h}E_{k]m} + 2NK_{m[k}E_{h]l} + 2H_{i[l}(\bar{\nabla}_{m]}N)\eta^i_{hk}. \end{aligned} \quad (31)$$

Analogously, from (19) we obtain

$$\hat{\partial}_0(\eta^j_{lm}B_{ij}) - N\eta^{kh}_i\eta^j_{lm}\bar{\nabla}_hD_{kj} + (L_4)_{i,lm} = -NJ_{i,lm}, \quad (32)$$

$$\begin{aligned} (L_4)_{i,lm} \equiv & -N(\text{Tr}\mathbf{K})\eta^j_{lm}B_{ij} + 2N\eta^h_{j[m}K^j_{l]}B_{ih} + 2NK^h_i\eta^j_{lm}B_{hj} \\ & - (\bar{\nabla}_jN)\eta^{hj}_i\eta^n_{lm}D_{hn} - 2N\eta^j_{hi}H_{j[m}K^h_{l]} + 2E_{i[m}\bar{\nabla}_{l]}N. \end{aligned} \quad (33)$$

Consider the system (30) and (32) with  $[lm]$  fixed. Then  $j$  in  $\eta_{jlm}$  is also fixed. The characteristic matrix for the  $[lm]$  equations, with unknowns  $D_{ij}$  and  $B_{ij}$ ,  $j$  fixed, with an orthonormal space frame, is the same as the matrix (28).

If the spacetime metric is considered as given, as well as the sources, the Bianchi equations (23), (26), (30), and (32) form a linear symmetric hyperbolic system with domain of dependence determined by the light cone of the spacetime metric. The coefficients of the terms of order zero are  $\bar{\nabla}N$  or  $N\mathbf{K}$ . The system is homogeneous in vacuum (zero sources).

## 4 Determination of $(\bar{\Gamma}, \mathbf{K})$ from Knowledge of the Bianchi Fields

We next link the metric and connection to our Bianchi fields. This link uses and extends an idea introduced by Friedrich [11] in his Weyl-tensor construction mentioned above. We will need the  $3 + 1$  decomposition of the Riemann tensor, which is [4]

$$R_{ij,kl} = \bar{R}_{ij,kl} + 2K_{i[k}K_{l]j}, \quad (34)$$

$$R_{0i,jk} = 2N\bar{\nabla}_{[j}K_{k]i}, \quad (35)$$

$$R_{0i,0j} = N(\hat{\partial}_0K_{ij} + NK_{ik}K^k_j + \bar{\nabla}_i\partial_jN). \quad (36)$$

From these formulae one obtains those for the Ricci curvature given in Sec. 2: (4), (5), and (6), where, in this section, we will *not* denote  $K^j_j = \text{Tr}\mathbf{K}$  by  $H$ . The identity (3) and the expression for the spatial Christoffel symbols give

$$\hat{\partial}_0 \bar{\Gamma}^h_{ij} \equiv \bar{\nabla}^h(NK_{ij}) - 2\bar{\nabla}_{(i}(NK_{j)}^h). \quad (37)$$

Therefore, from the identity (35), we obtain the identity

$$\hat{\partial}_0 \bar{\Gamma}^h_{ij} + N\bar{\nabla}^h K_{ij} = K_{ij}\partial^h N - 2K^h_{(i}\partial_{j)}N - 2R_{0(i,j)}^h. \quad (38)$$

On the other hand, the identities (36) and (4) imply the identity

$$\hat{\partial}_0 K_{ij} + N\bar{R}_{ij} + \bar{\nabla}_j \partial_i N \equiv -2NR^0_{i,0j} - N(\text{Tr}\mathbf{K})K_{ij} + NR_{ij}. \quad (39)$$

We obtain equations relating  $\bar{\Gamma}$  and  $\mathbf{K}$  to a double two-form  $\mathbf{A}$  and matter sources by replacing, in the identities (38) and (39),  $R_{0(i,j)}^h$  by  $(A_{0(i,j)}^h + A^h_{(j,i)0})/2$ ,  $R^0_{i,0j}$  by  $(A^0_{i,0j} + A^0_{j,0i})/2$ , and the Ricci tensor of spacetime by a given tensor  $\boldsymbol{\rho}$ , zero in vacuum. The terms involving  $\mathbf{A}$  are then replaced by Bianchi fields  $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{D}$  and  $\mathbf{B}$ .

The first set of identities (38) leads to equations with principal terms

$$\hat{\partial}_0 \bar{\Gamma}^h_{ij} + Ng^{hk}\partial_k K_{ij}. \quad (40)$$

To deduce from the second identity (39) equations which will form together with the previous ones a symmetric hyperbolic system, we set, we use algebraic gauge [8], as in Sec. 2, by setting

$$N = g^{1/2}\alpha \quad (41)$$

where  $\alpha$  is a given positive scalar density of weight minus one, a function of  $(t, x^i)$ . (Note that the present  $\alpha$  is the  $\alpha^{-1}$  of [16] and [17].) The lapse  $N$  is now a derived quantity depending on  $g^{1/2}$  and on  $\alpha$ . The use of  $\alpha$ , if  $\bar{\Gamma}$  denotes the Christoffel symbols of  $\mathbf{g}$ , implies that

$$\bar{\Gamma}^h_{ih} = \partial_i \log N - \partial_i \log \alpha. \quad (42)$$

The second set of identities (39) now yields the following equations, where  $N$  denotes  $g^{1/2}\alpha$ ,

$$\begin{aligned} \hat{\partial}_0 K_{ij} + N\partial_h \bar{\Gamma}^h_{ij} &= N[\bar{\Gamma}^m_{ih}\bar{\Gamma}^h_{jm} - (\bar{\Gamma}^h_{ih} + \partial_i \log \alpha)(\bar{\Gamma}^k_{jk} + \partial_j \log \alpha)] \\ &\quad - N(\partial_i \partial_j \log \alpha - \bar{\Gamma}^k_{ij}\partial_k \log \alpha) - N(E_{ij} + E_{ji}) - N(\text{Tr}\mathbf{K})K_{ij} + N\rho_{ij}. \end{aligned} \quad (43)$$

The first set (38) yields

$$\begin{aligned} \hat{\partial}_0 \bar{\Gamma}^h_{ij} + N \bar{\nabla}^h K_{ij} &= NK_{ij} g^{hk} (\bar{\Gamma}^m_{mk} + \partial_k \log \alpha) \\ &\quad - 2NK^h_{(i} (\bar{\Gamma}^m_{j)m} + \partial_j \log \alpha) - N(\eta^k_{(j} {}^h B_{i)k} + H_{k(i} \eta^k_{j)} {}^h). \end{aligned} \quad (44)$$

We see from the principal parts of (43) and (44) that the system obtained for  $\mathbf{K}$  and  $\bar{\Gamma}$  has a characteristic matrix composed of six blocks around the diagonal, each block a four-by-four matrix that is symmetrizable hyperbolic with characteristic polynomial  $N^4(\xi_0 \xi^0)(\xi_\alpha \xi^\alpha)$ . The characteristic matrix in a spatial orthonormal frame has blocks of the form

$$\begin{pmatrix} \xi_0 & N\xi_1 & N\xi_2 & N\xi_3 \\ N\xi_1 & \xi_0 & 0 & 0 \\ N\xi_2 & 0 & \xi_0 & 0 \\ N\xi_3 & 0 & 0 & \xi_0 \end{pmatrix}. \quad (45)$$

## 5 Symmetric Hyperbolic System for $(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{g}, \mathbf{K}, \bar{\Gamma})$

We denote by  $\mathbf{S}$  the system composed of the equations (23), (26), (30), (32), (3), (43), and (44), where the lapse function  $N$  is replaced by  $g^{1/2}\alpha$ . This system is satisfied by solutions of the Einstein equations whose shift  $\beta$ , hidden in the operator  $\hat{\partial}_0$ , has the given arbitrary values and whose lapse has the form  $N = g^{1/2}\alpha$ . (Clearly, any  $N > 0$  can be written in this form.) From the results of the previous sections, we see that for arbitrary  $\alpha$  and  $\beta$ , and given matter sources  $\rho$ , the system  $\mathbf{S}$  is a first order symmetrizable hyperbolic system for the unknowns  $(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{g}, \mathbf{K}, \bar{\Gamma})$ . Note that the various elements  $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{D}$ ,  $\mathbf{B}$ ,  $\mathbf{g}$ ,  $\mathbf{K}$ , and  $\bar{\Gamma}$  are considered as independent. For example, *a priori*, we neither know that  $\bar{\Gamma}$  denotes the Christoffel symbols of  $\mathbf{g}$  nor that  $\mathbf{E}$ ,  $\mathbf{H}$ ,  $\mathbf{D}$ , and  $\mathbf{B}$  are identified with components of the Riemann tensor of spacetime.

We now consider the *vacuum* case. The original Cauchy data for the Einstein equations are, with  $\phi$  a properly Riemannian metric and  $\psi$  a second rank tensor on an initial spacelike slice  $\Sigma_0$ ,

$$\mathbf{g}|_0 = \phi, \quad \mathbf{K}|_0 = \psi. \quad (46)$$

The tensors  $\phi$  and  $\psi$  must satisfy the constraints, which read in vacuum,

$$R_{0i} = 0, \quad (47)$$

$$\begin{aligned}
0 = G_{00} &\equiv R_{00} - \frac{1}{2}g_{00}R \\
&= R_{00} + \frac{1}{2}N^2g^{\alpha\beta}R_{\alpha\beta} \\
&= \frac{1}{2}N^2(\bar{R} - K_{ij}K^{ij} + (\text{Tr}\mathbf{K})^2),
\end{aligned} \tag{48}$$

with  $\bar{R} = g^{ij}\bar{R}_{ij}$ . The initial data given by  $\phi$  determine the Cauchy data  $\bar{\Gamma}^h_{ij}|_0$  and thus  $\bar{R}_{ij,kl}|_0$ . Then,  $R_{ij,kl}|_0$  is determined by using also  $\psi$ . To determine the initial values of the other components of the unknowns of the system  $\mathbf{S}$ , we use the arbitrarily given data  $\alpha$  and  $\beta$ . In particular, we use  $N = g^{1/2}\alpha$  to find  $R_{0i,jk}|_0$  and to compute  $\bar{\nabla}_j\partial_i N|_0$  appearing in the identity (4). We deduce from (4)  $\hat{\partial}_0 K_{ij}|_0$  when  $R_{ij} = 0$ , which enables  $R_{0i,0j}$  to be found from (36). All of the components of the Riemann tensor of spacetime are then known on  $\Sigma_0$ . We identify them with the corresponding components of the double two-form  $\mathbf{A}$  on  $\Sigma_0$ : the latter have thus *initially* the same symmetries as the Riemann tensor. We find the initial values of  $(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B})$  according to their definitions in terms of  $\mathbf{A}$ .

Detailed existence theorems and proofs that solutions of the EB system, with initial data given as in the preceding paragraph are found in [18]. Here we give a less detailed argument, which is nevertheless completely rigorous.

Let the initial data for the vacuum EB system be given as above. We know that our symmetrizable hyperbolic system has a unique solution. Because a solution of Einstein's equations in algebraic gauge [*i.e.*, with given  $\alpha(t, x)$ ]—previously proven to exist [5]—together with its connection and Riemann tensor, satisfies the present EB system while taking the same initial values, that solution must coincide with the solution of the present EB system in their common domain of existence.

The FOSH form of the ER system and the EB system are completely equivalent mathematical systems, with the EB system perhaps having its unknowns arranged in a more transparent way. For instance, the characteristic Bianchi fields with respect to a fixed space direction that propagate along the light cone are the directionally transverse components of the Riemann tensor. For a detailed discussion of characteristic fields of the FOSH form of the ER system—also curvatures—see [7]. These fields for the EB system will be discussed in more detail elsewhere, as will “third order” and “fourth order” forms of the EB system, analogous to those of the ER system.

## 6 A Concluding Remark About Constraint Violations

It is well known that the standard 3+1 equations (3)-(6), with time derivatives  $\hat{\partial}_0 g_{ij}$  and  $\hat{\partial}_0 K_{ij}$  only, were reformulated in terms of closely related variables and written in Hamiltonian form in far-reaching work by Arnowitt, Deser, and Misner [27] (ADM) and by Dirac [28]. Indeed, the standard 3+1 equations are often referred to as the “ADM equations.” However, it can be shown [29, 30] that when there are violations of the constraints (as there generically are in numerical work), the standard 3+1 evolution equations and the ADM evolution equations are not equivalent. From an analytic standpoint, the standard 3+1 equations in terms of  $(\mathbf{g}, \mathbf{K}, R_{ij}, R_{0i}, G^0_0)$  produce a first order symmetric hyperbolic system for the constraint violations, to which suitable “energy” bounds for the growth of these violations can be applied, while the actual ADM equations in terms of  $(\mathbf{g}, \boldsymbol{\pi}, G_{\mu\nu})$  do not produce a hyperbolic system for evolution of constraint violations. However, neither the standard 3+1 equations nor the ADM equations are known to be well-posed, independently of the issue of constraint conservation. The ER and EB systems are well-posed, as are the equations describing conservation of the constraints in these systems. This subject is treated in more detail elsewhere [31].

## Acknowledgment

J.W.Y. and A.A. were supported by National Science Foundation grants PHY-9413207 and PHY 93-18152/ASC 93-18152 (ARPA supplemented).

## References

- [1] Y. Choquet-Bruhat and J.W. York, in *General Relativity and Gravitation, I*, edited by A. Held (Plenum: New York, 1980), pp. 99-172.
- [2] A. Lichnerowicz, *Problèmes Globaux en Mécanique Relativiste*, (Hermann: Paris, 1939).
- [3] Y. Choquet (Foures)-Bruhat, J. Rat. Mechanics and Anal. **5**, 951-966 (1956).
- [4] J.W. York, in *Sources of Gravitational Radiation*, ed. L. Smarr, (Cambridge Univ. Press: Cambridge, 1979).

- [5] Y. Choquet-Bruhat and J. W. York, C. R. Acad. Sci. Paris, **t. 321**, Série I, 1089 (1995).
- [6] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, and J.W. York, Phys. Rev. Lett. **75**, 3377 (1995).
- [7] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, and J.W. York, Class. Quantum Grav. **14**, A9-A22 (1997).
- [8] Y. Choquet-Bruhat and T. Ruggeri, Commun. Math. Phys. **89**, 269-275 (1983).
- [9] C. Teitelboim, Phys. Rev. **D25**, 3159-2176 (1982); Phys. Rev. **D28**, 297-309 (1983).
- [10] A. Ashtekar, *New Perspectives in Canonical Gravity*, (Bibliopolis: Naples, 1988).
- [11] H. Friedrich, Class. Quantum Grav. **13**, 1451-1459 (1996).
- [12] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, and J.W. York, to appear in Proceedings of the 1996 Texas Symposium on Relativistic Astrophysics, gr-qc/9703010 (1997).
- [13] J. Leray, *Hyperbolic Differential Equations*, lecture notes, Institute for Advanced Study, Princeton (1952).
- [14] A. Fischer and J. Marsden, Commun. Math. Phys. **28**, 1-38 (1972).
- [15] S. Frittelli and O. Reula, Commun. Math. Phys. **166**, 221-235 (1994); Phys. Rev. Lett. **76**, 4667-4670 (1996).
- [16] Y. Choquet-Bruhat and J. W. York, Banach Center Publications, **41**, Part 1, 119-131 (1997).
- [17] A. Anderson, Y. Choquet-Bruhat and J.W. York, to appear in Topological Methods in Nonlinear Analysis, gr-qc/9710041, (1997).
- [18] Binary Black Hole Grand Challenge Alliance: M. Rupright, A. Abrahams, L. Rezzolla, *et al.*, to appear in Phys. Rev. Lett., gr-qc/9709082, (1997); M. Rupright, A. Abrahams, and L. Rezzolla, gr-qc/9802011 (1998).
- [19] Binary Black Hole Grand Challenge Alliance: R. Gomez, L. Lehner, R. Marsa, J. Winicour, *et al.*, gr-qc/9801069 (1998).

- [20] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, and J.W. York, C.R. Acad. Sci. Paris, **t. 323**, Série II *b*, 835-841 (1996).
- [21] J. Leray and Y. Ohya, Math. Ann. **170**, 167-205 (1967).
- [22] Y. Choquet-Bruhat, in *Modern Group Analysis*, eds. N. Ibragimov and F. Mahomed (World Scientific: Singapore), to appear (1997).
- [23] Y. Choquet-Bruhat and A. Greco, Circolo Matematico di Palermo, to appear (1997).
- [24] A. Anderson, A. Abrahams, and C. Lea, gr-qc/9801071 (1998).
- [25] L. Bel, C. R. Acad. Sci. Paris **246**, 3105 (1958) and *Thesis*, University of Paris (1961).
- [26] C. Lanczos, Ann. of Math. **39**, 842-850 (1938).
- [27] R. Arnowitt, S. Deser, and C.W. Misner, in *Gravitation: An Introduction to Current Research*, ed. L. Witten, (Wiley: New York, 1962), pp. 227-265.
- [28] P.A.M. Dirac, Proc. Roy. Soc. London **A246**, 333-343 (1958).
- [29] S. Frittelli, Phys. Rev. **D55**, 5992-5996 (1997).
- [30] A. Anderson and J.W. York, unpublished.
- [31] A. Anderson and J.W. York, "Propagation of constraint violations in Hamiltonian formulations of general relativity," in preparation.